APPENDIX F

SELECTED MATHEMATICAL TOPICS

1. Exponents ................................................. F-2
   a. Exponential Equations .............................. F-4
   b. Power Functions .................................... F-5
2. Common Logarithms (Base 10) ............................ F-6
3. Natural or Napierian Logarithms .......................... F-8
4. Equations ................................................ F-10
   a. Linear Equations .................................... F-11
   b. Quadratic Equations ................................. F-12
5. Proportionality .......................................... F-14
6. Geometry ............................................... F-16
7. Trigonometry ........................................... F-19
1. Exponents

On occasion, it is necessary to multiply a number by itself several times. This can be cumbersome to write out if a large number of such multiplications must be accomplished. In order to simplify writing this, these expressions are often written in exponential form. This consists of expressing the number, or quantity in the case of an algebraic symbol, with a superscript to the right of that number or quantity. This superscript is called the exponent (or power), and expresses the number of times the number or quantity (called the base) is multiplied by itself. For example, let $B \cdot B \cdot B$ be the quantity which is multiplied by itself. In exponential notation, this is written

$$B \cdot B \cdot B = B^3$$

If the quantity $B \neq 0$, then $B^0 = 1$ by definition.

When two of the same quantity, each raised to a certain power, are multiplied, the resultant power equals the sum of the exponents or powers. That is,

$$B^n \cdot B^m = B^{n+m}$$

To extend this, if a a given quantity which is raised to a certain power ($B^n$) is multiplied by itself $m$ times, then

$$(B^n)^m = B^{nm}$$

For example,

$$(B^2)^4 = B^2 \cdot B^2 \cdot B^2 \cdot B^2 = B^8$$
This rule may also be extended to multiplication of different quantities with the same exponent:

\[(B^n)(C^n) = (BC)^n\]

A quantity or number which has a minus sign in the exponent signifies the reciprocal of that quantity to the plus power:

\[1/B^2 = B^{-2}\]

The rules for combining quantities with negative powers, or for combining negative with positive powers, are the same as for combining those with positive exponents,

\[B^{-n}B^{-m} = B^{-(n+m)}\]

\[B^nB^{-m} = B^n/B^m = B^{n-m}\]

\[(B^{-n})^{-m} = B^{nm}\]

A quantity can be expressed with a fractional exponent. This occurs when one quantity raised to a power equals another quantity

\[B^n = K\]

in which \(n\) is a positive integer. The quantity \(B\) is called the \(n\)th root of \(K\). The above equation can be written in terms of a fractional exponent:

\[B = K^{1/n}\]

For example,
\[
K^{5/2} = \sqrt{K^5}
\]
\[
2^{1/2} = \sqrt{2}
\]
\[
3^{2/3} = 3\sqrt[3]{3^2}
\]

a. **Exponential Equations**

An exponential equation is one in which the unknown appears as an exponent. The exponential is the value which results when a given number, called the base, is raised to some exponent, \(x\). In the Napierian system, the quantity \(e\) is used as the base. The value of the base \(e\) to six significant figures is 2.71828.

Exponential tables using the Napierian base give the value of \(e^x\) or \(e^{-x}\). These tables are used in solving problems in which the rate of change of one quantity with respect to another is a constant fraction. Applications such as radioactive decay and radiation attenuation involve the use of these tables.

In general, the exponential equation has the form

\[
y = e^x \text{ or } y = e^{-x}.
\]

In the case of radioactive decay, we write

\[
\frac{N}{N_0} = e^{-\lambda t} \text{ or } \frac{A}{A_0} = e^{-\lambda t},
\]

in which \(\frac{N}{N_0}\) or \(\frac{A}{A_0}\) stands for \(y\), and \(\lambda t\) stands for \(x\). Similarly, for attenuation problems the form becomes

\[
\frac{I}{I_0} = e^{-\mu x},
\]

where

\[
\frac{I}{I_0}
\]

now stands for \(y\) and the product \(\mu x\) replaces \(x\).
Example: A radioactive sample has a decay constant \( \lambda = 0.1 \text{d}^{-1} \).Given the present activity \( A_0 = 600 \text{ dis/min} \), find theactivity 12 days later.

\[
\frac{A}{600} = e^{-\lambda t} = e^{-(0.1)12} = e^{-1.2}.
\]

From five-place tables, the value 0.30119 is given for \( e^{-1.2} \),then,

\[
\frac{A}{600} = 0.30119 \text{ and } A = (600)(0.30119) = 181 \text{ dis/min}.
\]

Assume that we were asked the activity at time \( t = 12 \) days earlier. In this case, \( A = 600 \text{ dis/min} \) and we write

\[
\frac{600}{A_0} = e^{-\lambda t} \text{ or }
\]

\[
\frac{A_0}{600} = e^{\lambda t}.
\]

Again \( x = \lambda t = 1.2 \), so that we locate \( x = 1.2 \) in thefive-place table and read the value 3.3201 under the \( e^x \) column.Then,

\[
\frac{A_0}{600} = 3.3201 \text{ and } A_0 = (600)(3.3201) = 1992 \text{ dis/min}.
\]

With the increased use of calculators, the use of exponential tables hasgiven way to use of calculators. If the calculator has an \( e^x \) key, one simply enters the value of \( x \) (positive or negative) and presses the \( e^x \) key to obtain the value of \( e^x \) in the display. For calculators with theINV \( \ln x \) keys, one enters the value of \( x \) and presses the INV \( \ln x \) keys to obtain the value of \( e^x \) in the display.

b. **Power Functions**

A number of exponential equations involve a base which is not
the quantity \( e \). These equations are referred to as power function relationships. These are generally expressed by

\[
z = ay^x,
\]

in which \( a \) and \( x \) are constants, \( y \) and \( z \) are variables. In this case, \( y \) is a variable base. The exponential quantity \( y^x \) is easily solved on present day calculators. One simply enters the value \( y \) and presses the \( y^x \) key. Then enter the exponent value \( x \) and press the \( = \) key to have the quantity evaluated in the display. In addition, most calculators have a separate \( x^2 \) key for use when one is simply squaring a number \( x \) and a \( \sqrt{x} \) key for finding the square root of a given number \( x \).

2. **Common Logarithms (Base 10)**

A common logarithm is the exponent which expresses the power to which the base 10 must be raised in order to equal a given number. The exponent is called the common logarithm of that number. The common logarithm consists of the **characteristic** (the figures to the left of the decimal point) and the **mantissa** (the figures to the right of the decimal point).

The characteristic gives the order of magnitude of the given number, both by the figure and its algebraic sign. A positive figure indicates the number of factors of 10 and a negative figure, the number of factors of \( 1/10 \). A zero characteristic denotes a number which lies between one and 10. The mantissa gives the significant figures in the logarithm of the given number without regard to the decimal point placement. To find the common logarithm using logarithm tables, convert the number to scientific notation. For example, let \( N = 0.765 \). Find the common logarithm of \( N \). Write \( N \) as \( \text{N} = 7.65 \times 10^{-1} \). The mantissa of the logarithm can be found from a suitable table, by first locating 76 under the \( N \) column and reading 88377 in the S column immediately to the right of 76 in the same row. The characteristic is \( 1 \) for 0.765. Note that for a number with n
digits to the left of the decimal, the characteristic of the logarithm of that number is $n-1$. For a positive number less than one, the characteristic is $\bar{n}$, if the first digit (other than zero) appears in the $n$th place to the right of the decimal. The common logarithm (often written $\log_{10}$ or simply $\log$) of 0.765 is then $1.8837$. If the number were 7.65, then $\log_{10} 7.65 = 0.8837$. If the number were 76.5, $\log_{10} 76.5 = 1.8837$. Note that the algebraic sign applies only to the characteristic, not the mantissa.

As an alternate notation, a logarithm with a negative characteristic may be changed to one with a positive characteristic. A common method is to write the logarithm as

$$\log_{10} 0.765 = 0.8837 - 1 = -0.1163.$$ 

This notation is used on calculators to indicate numbers with a negative characteristic. When one enters the number 0.765 and presses the log key on the calculator, the display will indicate -0.1163 for the first four places.

Properties which make logarithms useful are:

\[
\begin{align*}
\log_{10} (ab) &= \log_{10} a + \log_{10} b \\
\log_{10} (a/b) &= \log_{10} a - \log_{10} b \\
\log_{10} (1/a) &= \log_{10} 1 - \log_{10} a = 0 - \log_{10} a \\
\log_{10} a^n &= n \log_{10} a \\
\log_{10} a^{-n} &= -n \log_{10} a \\
\log_{10} \sqrt[n]{a} &= \frac{1}{n} \log_{10} a
\end{align*}
\]

The antilog is the given number that the logarithm represents. From our previous example for 76.5:

$$\log_{10} 76.5 = 1.8837.$$
The antilog of 1.8837 is the given number 76.5. To find a number when given the logarithm of the number is the inverse of the previous process for finding the logarithm.

In the antilog process, locate the mantissa of the logarithm in a common logarithm table. The first two significant figures are found in the same row as the mantissa, but to the left under the N column. The third significant figure is the number at the top of the column in which the mantissa is located. Given the common logarithm 2.6064, find the antilog to three significant figures: In a logarithm table find 6064, read 40 to the left under the N column as the first two figures. In the same column as 6064, read 4 at the top of the column for the third figure. The antilog is then 404, exclusive of the decimal point placement.

The characteristic of the logarithm allows one to locate the decimal point. In our example, the characteristic is positive so that the number of figures to the left of the decimal point will be one more than the value of the characteristic. That is, for a characteristic of +2, there will be three figures to the left of the decimal point. The final answer for the antilog in our example is then

\[ \text{antilog } 2.6064 = 404. \]

On a calculator the procedure is to enter the logarithm (2.6064), and press the INV and log keys and the antilogarithm will be shown in the display.

3. Natural or Napierian Logarithms

A natural or Napierian logarithm is the exponent which expresses the power to which the base \( e \) must be raised to equal a given number. The exponent is called the natural logarithm (often written \( \ln \) or \( \log_e \)) of the number. If the given number is less than unity, the logarithm is negative. If the number is greater than unity, the logarithm is positive. The logarithm of unity is zero.
Natural logarithm tables contain the natural logarithms of numbers from 1.00 to 9.99. Logarithms of smaller or larger numbers with the same significant figures are found by use of the property

\[ \ln A = \ln(N \times 10^k) = \ln N + k \ln 10, \]

in which \( A \) is a number larger than 9.99 or smaller than 1.00, \( N \) is a number with the same significant figures as \( A \) but lies between 1.00 and 9.99, and \( k \) is the power of ten by which \( N \) must be multiplied to equal \( A \). Included with the table are the natural logarithms of \( 10^k \), up to \( k = 9 \).

To use the table to find the natural logarithm of 3.97, look up the first two digits under the column marked \( N \). That is, find 3.9 in the \( N \) column. Along the same row to the right of 3.9, the logarithm is read as 1.3788 under column 7. On a calculator with a \( \ln x \) key, one simply enters the value of the number for which the natural logarithm is to be found. The logarithm is then found by pressing the \( \ln x \) key. The calculator may display the logarithm to more digits than are found in the logarithm tables.

The properties listed in F.2 for common logarithms apply also to natural logarithms. Actually, the logarithm properties may be applied to any base.

To find the logarithm of a number to a different base, the following may be used

\[ \log_x N = (\log_y N)(\log_x y). \]

For change of base involving 10 and \( e \), the above expression requires the use of these relationships

\[ \log_{10} e = 0.43429 \]

\[ \ln 10 = 2.3026 \]
4. **Equations**

An equation expresses that a given quantity is equal to some other quantity. An arithmetic equation involves the use of numbers; for example

\[ 4 + 5 = 9. \]

In this, the quantity to the left of the equal sign is equal to the quantity on the right of the sign. When the equation contains one or more unknowns, which are designated by symbols, the equation is called an algebraic equation. For example,

\[ 2x + 8 = 18 \]

is an algebraic equation in which the symbol \( x \) represents the unknown value. Not all values of \( x \) will satisfy the above equality. Finding all the possible values of the unknown which satisfy the above relationship is referred to as solving the equation. For some equations, called identities, all values of the unknown are solutions. For other equations, as in the above example, only specific values are valid solutions. In the example, the solution \( x=5 \) is the only value for which the equation is true.

In approaching algebraic equations, the following operations may be useful:

\[
\begin{align*}
    a + b &= b + a \\
    ab &= ba \\
    (a+b)+c &= a + (b + c) \\
    (ab)c &= a(bc) \\
    a + o &= a \\
    a.o &= o
\end{align*}
\]
Certain principles which govern the manipulations used to solve algebraic equations are:

1. An operation performed on one side of the equation must be performed on the other side.

2. A term may be transposed from one side of the equation to the other side by changing the sign of the term:

   \[ a + b = c \]
   \[ a = c - b \]

   In the example, the transposition of \( b \) is equivalent to the operation of adding \(-b\) to each side of the equation.

3. A quantity used as a multiplier on one side of the equation may be transposed as a divider to the other side of the equation, and vice versa. For example,

   \[ bc = d \]
   \[ c = d/b \]

   This maneuver corresponds to the operation of dividing each side of the equation by \( b \).

a. **Linear Equations**

   An algebraic equation in which the unknown(s) are to the first power is called a linear equation. A frequent type of linear equation encountered describes the relationship between two unknowns \( x \) and \( y \). This is the linear equation of a straight line:

   \[ y = a + bx. \]
This form of the straight line equation is referred to as the slope-intercept form. The constant \( a \) is called the intercept, which is the value of \( y \) when \( x = 0 \). The slope \( b \) is a measure of the relative change in \( y \) values for a change in the \( x \) values. It is expressed as the ratio of the difference in the two \( y \) values to the corresponding difference in the \( x \) values, 
\[
b = \frac{y_1 - y_0}{x_1 - x_0},
\]
in which \( x_1, y_1 \) and \( x_0, y_0 \) are the two points. A special case of the linear equation is a straight line through the origin. For this case, \( a = 0 \).

Some equations, such as radioactive decay or exponential attenuation, which are not straight lines can be put into the form of a straight line for ease in graphing. For example, consider the activity equation

\[
A = A_0 e^{-\lambda t}
\]

Take the natural logarithm of each side of the equation. This gives

\[
\ln A = \ln A_0 - \lambda t,
\]

since \( \ln e = 1 \). Now, if one considers \( \ln A = y \), \( \ln A_0 = a \), \( t \) as the unknown \( x \), then \( \lambda \), the transformation constant, is \( b \), the slope of the line. The negative sign indicates that the variable \( y \) (\( \ln A \)) decreases with increases in the variable \( x \) (\( t \)). By graphing on semilog paper (see Section 2.c), the plot will be a straight line.

b. **Quadratic Equations**

In a quadratic equation, the highest power of any unknown is 2. This type of equation is expressed by the general form

\[
ax^2 + bx + c = 0,
\]

in which \( a, b \) and \( c \) are constants and \( a \neq 0 \). The process of solving a quadratic equation is called finding the roots of the equation.
If the linear term in \( z \) is missing, the equation then reduces to an expression for the square of the unknown. Consider

\[
\begin{align*}
12 \ z^2 - 768 &= 0 \\
12 \ z^2 &= 768 \\
\ z^2 &= 64 \\
\ z &= \sqrt{64} = \pm 8
\end{align*}
\]

Sometimes the left side of the equation can be factored into the product of several simpler expressions. When this can be accomplished, the result is an equation in which a number of quantities (or factors) are multiplied together to give the product zero. Thus, any one or all of the factors could be zero. Solutions are obtained by setting each factor equal to zero and solving for the value of \( z \). For example, solve

\[
\ z^2 + 6z + 8 = 0.
\]

The expression on the left side of the equality sign can be factored into the product \((z + 4)(z + 2)\), so that the equation may be written

\[
(z + 4)(z + 2) = 0.
\]

Setting each factor equal to 0 and solving gives

\[
\ z + 4 = 0, \ z = -4 \\
\ z + 2 = 0, \ z = -2.
\]

When factoring of the equation is not possible, the quadratic formula can be used to solve for the roots. In fact, it can be used to find the roots of the equation, regardless of whether one can find factors or not. The quadratic formula is
\[ z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

One root is found using the positive value of the square root of \( b^2 - 4ac \), and one is found using the negative value.

5. **Proportionality**

When one finds that the value of one quantity is related to the value of another quantity, such that if the one is increased or decreased by a certain factor, the other also increases, or decreases, by the same factor, the two quantities are said to be directly proportional to one another.

In Section 1.6, some examples of direct proportions are presented. These include Ohm's Law, which states that the current in a circuit is directly proportional to the applied voltage, and the relationship between charge buildup on a capacitor and the applied voltage. In each case, the relation can be transformed into an equation by specifying a constant of proportionality. Consider a capacitor, then

\[ Q \propto V, \]

in which \( Q \) is the symbol for the charge which builds up on the plates of the capacitor, \( \propto \) is the symbol used to indicate proportionality and \( V \) is the applied voltage. To transform this expression into an equation, \( C \), which is called the capacitance, is specified as the constant of proportionality, and the relationship becomes equation 1.16,

\[ Q = CV \]

Note that this equation has the same form as that of a straight line through the origin, with \( C \) being the slope of the line. Other direct proportions may involve powers of the quantities, such as \( A = 2\pi r^2 \), or roots, such as \( r = 1.3 \times 10^{-15} \text{ A}^{1/3} \).
In a statement of direct proportion between two quantities, it is not always necessary to know the value of the constant of proportionality in order to use the relationship. For example, the radius of a circle is directly proportional to the circumference, so that

\[ C \propto r \]
\[ C = kr. \]

Given two circles, one with \( C_1 = 31.416, \ r_1 = 5 \), find \( C_2 \), if \( r_2 = 8 \). This may be solved by forming ratios of the quantities:

\[ \frac{C_2}{C_1} = \frac{k \ r_2}{k \ r_1} = \frac{r_2}{r_1} \]

since \( k \), the constant of proportionality, is always the same value. Substituting values gives

\[ \frac{C_2}{31.416} = \frac{8}{5} \]

\[ C_2 = 1.6(31.416) = 50.266. \]

If two quantities are inversely proportional to each other, then an increase by some factor of one quantity results in a corresponding decrease in the other quantity. This may be expressed as

\[ A \propto 1/B \]
\[ A = C/B, \]

in which \( C \) represents the constant of proportionality. As in the case of direct proportions, an inverse proportionality may involve higher powers of the quantities. Such an example is the inverse square law (Section 3.D). For a point source of radiation, the intensity at point \( A \), a distance \( r_A \) from a source may be written
\[ I_A \propto \frac{1}{r_A^2} = \frac{k}{r_A^2}, \]

in which \( k \) represents the constant of proportionality. At point \( B \), a distance \( r_B \) from the point source, the intensity will be

\[ I_B = \frac{k}{r_B^2}. \]

Taking the ratio of these two expressions eliminates \( k \). This allows one to use the known values of the intensity at some distance from the source to compute the unknown intensity at a different distance from the source, using

\[ \frac{I_A}{I_B} = \frac{r_B^2}{r_A^2}. \]

which is equation 3.26d in the text.

6. Geometry

Several concepts of plane and solid geometry are useful in health physics applications, particularly those dealing with angles, areas and volumes.

Angles may be expressed in degrees or radians. A right angle is a plane angle of 90°, a circle contains 360°, and the sum of the interior angles of a triangle is 180°. Subdivisions of the degree are minutes and seconds. There are 60 seconds (60") in 1 minute (1') and 60 minutes (60') in one degree (1°).

Angles may also be expressed in terms of radians. In the figure, the angle \( \theta \), in radians, is equal to the arc length \( s \),

\[ \begin{array}{c}
\theta \\
\frac{s}{r}
\end{array} \]

subtended by the angle, divided by the radius \( r \) of the circle, or
\[ \theta = \frac{s}{r}. \]

Since the total arc length around the circle is the circumference, which is equal to \(2\pi r\), the total angle, in radians, in a circle is

\[ \theta_{\text{total}} = \frac{2\pi r}{r} = 2\pi \text{ radians}. \]

The relationship between radians and degrees is then

\[ 360^\circ = \theta_{\text{total}} = 2\pi \text{ radians} \]

\[ 1^\circ = \frac{2\pi}{360} \text{ radians} \]

Several useful expressions for the area and the perimeter of plane figures are:

1. **Square**
   
   ![Square](image)
   
   Area = \(a^2\), perimeter = \(4a\)

2. **Rectangle**

   ![Rectangle](image)
   
   Area = \(ab\), perimeter = \(2a + 2b\)

3. **Circle**

   ![Circle](image)
   
   Area = \(\pi r^2\), circumference = \(2\pi r\)

4. **Triangle**

   ![Triangle](image)
   
   Area = \(1/2 \ ab\), perimeter = \(b+c+d\)
Plane geometry is concerned with figures that have at most, two dimensions. Solid geometry is concerned with figures that have up to three dimensions. Several expressions of use for certain common geometric solids are:

1. **rectangular solid**
   
   ![Rectangular Solid Diagram](image)
   
   Area = $2xz + 2yz + 2xy$
   
   Volume = $xyz$

2. **cylinder**
   
   ![Cylinder Diagram](image)
   
   Area = $2\pi rh + 2\pi r^2$
   
   Volume = $\pi r^2 h$

3. **sphere**
   
   ![Sphere Diagram](image)
   
   Area = $4\pi r^2$
   
   Volume = $4/3\pi r^3$

One other important concept from solid geometry, of use in health physics, is the solid angle, $\Omega$. The solid angle is measured by the ratio of the surface area $A$, of the portion of a sphere enclosed by a cone which forms the solid angle $\Omega$ (see figure), to the square of the radius $R$ of the sphere:

![Solid Angle Diagram](image)

\[ \Omega = \frac{A}{R^2} \]

The unit of solid angle is the steradian, sr. The total solid angle $\Omega_{total}$ in a sphere is given by:

\[ \Omega_{total} = \frac{A}{R^2} = \frac{4\pi R^2}{R^2} = 4\pi \text{sr.} \]
The concept of the fractional solid angle, called the geometry (see Section 11.H.4), is used to determine the fraction of radiation emitted from an isotropic point source which reaches the detector.

7. **Trigonometry**

Trigonometric functions are defined in terms of the ratios of the sides of a right triangle. A right triangle contains an angle of 90°, called a right angle. In order to have a right angle, one of the legs of a triangle must be perpendicular to one of the other legs of the triangle. In the figure, the 90° angle is C and the side c of the triangle opposite the right angle is called the hypotenuse.

The functions of angle A defined by the ratios of the sides are:

\[
\text{sine } A = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}} \quad \text{cosecant } A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}
\]

\[
\text{cosine } A = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}} \quad \text{secant } A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}}
\]

\[
\text{tangent } A = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}} \quad \text{cotangent } A = \frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}}
\]

The trigonometric functions are useful for solving for unknown sides or angles in the triangle by utilizing the given information concerning the rest of the triangle. Values for the sine, cosine and tangent of a given angle can be obtained on calculators which have the keys sin, cos, and
tan, respectively. The cofunctions listed above are simply the reciprocal of the function value. In addition, given a right triangle in which the value of the sine is known, the angle can be found by entering the value of the sine and pressing the INV sin keys.

One other useful relationship regarding right triangles is the Pythagorean theorem. Using the figure above, this relationship is

\[ a^2 + b^2 = c^2, \]

or the square of the hypotenuse equals the sum of the squares of the other sides.

Two properties of the trigonometric functions which apply to any triangle are called the law of sines and the law of cosines. Using the terminology of the previous figure, the law of sines may be written:

\[ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \]

in which sin is used as an abbreviation for sine.

Using the abbreviation cos for cosine, the law of cosines may be stated:

\[ a^2 = b^2 + c^2 - 2bc \cos A. \]

If one knows the value of two sides and the included angle of a triangle, the other side can be determined. For example, given the two sides and the included angle in the figure, solve for the value of side a:

\[ a^2 = b^2 + c^2 - 2bc \cos A \]
\[ = (6)^2 + (4)^2 - 2(6)(4) \cos 60^\circ \]
\[ a^2 = 28 \]
\[ a = \pm \sqrt{28} = \pm 5.29 \]

In this case, it is clear from the physical picture that a=5.29 is the solution.
REFERENCES
